COL7160 : Quantum Computing

Lecture 3: Measurement

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1 Properties of Tensor Products

Before discussing measurement, we explicitly state and prove a fundamental property regarding the inner product of tensor product states.

Lemma 1. *Let $\{|e_i\rangle\}_{i=1}^m$ be the standard basis for $\mathbb{C}^m$ (where $|e_i\rangle$ has a 1 at the i-th position and 0 elsewhere) and $\{|f_j\rangle\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. The set of tensor products $B = \{|e_i\rangle \otimes |f_j\rangle : 1 \leq i \leq m, 1 \leq j \leq n\}$ forms an orthonormal basis for $\mathbb{C}^{mn}$ under the standard inner product.*

Proof. By definition of the Kronecker product, the vector $|e_i\rangle \otimes |f_j\rangle$ is a vector of length mn that has exactly one non-zero entry (equal to 1) at the index corresponding to the block i and offset j. Distinct pairs $(i, j) \neq (k, l)$ result in vectors with 1s at strictly different positions. Therefore, the standard inner product (dot product) between any two distinct basis vectors is 0:

$$\langle e_i \otimes f_j | e_k \otimes f_l \rangle = 0 \quad \text{if } (i, j) \neq (k, l)$$

If $(i, j) = (k, l)$, the inner product is $1 \times 1 = 1$. Thus, $\langle e_i \otimes f_j | e_k \otimes f_l \rangle = \delta_{ik}\delta_{jl}$, proving orthonormality. Since there are mn such vectors in an mn-dimensional space, they form a basis. □

Proposition 2. *Let $|\psi_0\rangle, |\psi_2\rangle \in \mathbb{C}^m$ and $|\psi_1\rangle, |\psi_3\rangle \in \mathbb{C}^n$. The inner product of the tensor product states factorizes into the product of individual inner products:*

$$(\langle\psi_0| \otimes \langle\psi_1|) \cdot (|\psi_2\rangle \otimes |\psi_3\rangle) = \langle\psi_0|\psi_2\rangle \cdot \langle\psi_1|\psi_3\rangle$$

Proof. We expand the arbitrary vectors in the standard bases defined in the Lemma:

$$|\psi_0\rangle = \sum_i a_i |e_i\rangle, \quad |\psi_2\rangle = \sum_k b_k |e_k\rangle$$

$$|\psi_1\rangle = \sum_j c_j |f_j\rangle, \quad |\psi_3\rangle = \sum_l d_l |f_l\rangle$$

The inner product on the combined space $\mathbb{C}^{mn}$ is linear. Substituting the expansions into the LHS:

$$\begin{aligned} \text{LHS} &= \left(\sum_{i,j} a_i^* c_j^* (\langle e_i| \otimes \langle f_j|) \right) \cdot \left(\sum_{k,l} b_k d_l (|e_k\rangle \otimes |f_l\rangle) \right) \\ &= \sum_{i,j,k,l} a_i^* c_j^* b_k d_l \langle e_i \otimes f_j | e_k \otimes f_l \rangle \end{aligned}$$

Using the result from the Lemma, $\langle e_i \otimes f_j | e_k \otimes f_l \rangle = \delta_{ik}\delta_{jl}$. This collapses the sum to indices where $i = k$ and $j = l$:

$$\begin{aligned} &= \sum_{i,j} a_i^* c_j^* b_i d_j \\ &= \left(\sum_i a_i^* b_i \right) \cdot \left(\sum_j c_j^* d_j \right) \\ &= \langle\psi_0|\psi_2\rangle \cdot \langle\psi_1|\psi_3\rangle \end{aligned}$$

□

2 Measurement

Measurement in quantum mechanics is a fundamental operation that differs significantly from classical observation. It is a **probabilistic** and **irreversible** operation. To perform a measurement, we must specify an orthonormal basis corresponding to the property we wish to observe.

2.1 Measurement in the Computational Basis

Consider a single qubit in the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$. If we measure this qubit in the standard computational basis $\{|0\rangle, |1\rangle\}$:

- We observe the output **0** with probability $|\alpha|^2$. The state collapses to $|0\rangle$.
- We observe the output **1** with probability $|\beta|^2$. The state collapses to $|1\rangle$.

This illustrates how randomness can emerge from a deterministic state.
Suppose we prepare a qubit in the superposition state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. If we measure this state in the computational basis $\{|0\rangle, |1\rangle\}$:

- Probability of 0: $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.
- Probability of 1: $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.

Even though the state $|+\rangle$ was prepared deterministically, the measurement outcome is random.

2.2 Measurement in a General Basis

We are not restricted to the computational basis. We can measure a qubit in any orthonormal basis $B = \{|b_0\rangle, |b_1\rangle\}$ of $\mathbf{C}^2$. To find the probabilities, we express the state $|\psi\rangle$ as a linear combination of the basis vectors:

$$|\psi\rangle = \alpha' |b_0\rangle + \beta' |b_1\rangle$$

where $\alpha' = \langle b_0|\psi\rangle$ and $\beta' = \langle b_1|\psi\rangle$.
The measurement rules are:

- The system collapses to $|b_0\rangle$ with probability $|\alpha'|^2 = |\langle b_0|\psi\rangle|^2$.
- The system collapses to $|b_1\rangle$ with probability $|\beta'|^2 = |\langle b_1|\psi\rangle|^2$.

3 Global Phase and Distinguishability

A common question arises regarding the physical significance of the phase of a quantum state. Specifically, can we distinguish between a state $|\psi\rangle$ and the state $e^{i\theta} |\psi\rangle$ (where $\theta \in [0, 2\pi]$) using measurement?
To answer this, consider an arbitrary quantum system (not necessarily a single qubit) described by a state vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. Let us perform a measurement in an arbitrary orthonormal basis $\{|b_k\rangle\}_k$.
The probability of obtaining the outcome corresponding to the basis vector $|b_k\rangle$ given the state $|\psi\rangle$ is:

$$P(k) = |\langle b_k|\psi\rangle|^2$$

Now, consider the state with a global phase, $|\psi'\rangle = e^{i\theta} |\psi\rangle$. If we perform the same measurement on this state, the probability of obtaining the outcome k is:

$$P'(k) = |\langle b_k|\psi'\rangle|^2 = |\langle b_k|e^{i\theta}\psi\rangle|^2$$

Using the linearity of the inner product in the second argument (or simply factoring out the scalar), we have:

$$\langle b_k|e^{i\theta}\psi\rangle = e^{i\theta} \langle b_k|\psi\rangle$$

Taking the squared modulus:

$$P'(k) = |e^{i\theta} \langle b_k|\psi\rangle|^2 = |e^{i\theta}|^2 |\langle b_k|\psi\rangle|^2$$

Since $|e^{i\theta}| = 1$ for any real θ, we arrive at:

$$P'(k) = 1 \cdot |\langle b_k|\psi\rangle|^2 = P(k)$$

Since the measurement probabilities for all possible outcomes in any basis are identical for $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$, there is no physical measurement that can distinguish between them.

Theorem 3. *The states $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ are physically indistinguishable. The factor $e^{i\theta}$ is called a **global phase** and has no observable consequences.*

Remark 4. Note that relative phase *does* matter. For example, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ are distinguishable (e.g., by measuring in the Hadamard basis).

3.1 Distinguishing Non-Orthogonal States

If we are given an unknown state $|\psi\rangle$ promised to be from the set $\{|\varphi_1\rangle, |\varphi_2\rangle\}$, can we identify which one it is?

- If $\langle\varphi_1|\varphi_2\rangle = 0$ (orthogonal), we can distinguish them perfectly by measuring in a basis containing $|\varphi_1\rangle$ and $|\varphi_2\rangle$.

Exercise: Suppose the states are not orthogonal, i.e., $\langle\varphi_1|\varphi_2\rangle \neq 0$. Find the maximum probability with which these two states can be distinguished using an optimal measurement.

4 Interaction-Free Measurement: The Bomb Test

This thought experiment, known as the Elitzur–Vaidman bomb tester, demonstrates the power of quantum measurement to detect the presence of an object without interacting with it (i.e., without a photon hitting it).

4.1 The Setup

Consider a Mach-Zehnder interferometer with two paths, labeled $|0\rangle$ and $|1\rangle$. A bomb is placed on the path corresponding to the state $|1\rangle$. The bomb has two possible states:

1. **Working (Live):** The bomb is light-sensitive. If a photon hits it (path $|1\rangle$), it explodes. In quantum mechanical terms, a live bomb acts as a **measurement device** in the computational basis $\{|0\rangle, |1\rangle\}$.

2. **Not Working (Dud):** The bomb is broken (transparent). The photon passes through without any interaction. This acts as the identity operator $\mathbb{I}$.

Our goal is to determine if the bomb is working *without* detonating it.

4.2 The Protocol

The experiment proceeds in two distinct measurement stages:

1. **Input:** We inject a photon. While we could use a general state $|\psi\rangle$, for simplicity we prepare the equal superposition state:

$$|\psi_{in}\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

2. **Stage 1 (The Bomb):** The photon passes the bomb.
   - If the bomb is **Live**, it measures the photon's position in the basis $\{|0\rangle, |1\rangle\}$.
   - If the bomb is a **Dud**, no measurement occurs.

3. **Stage 2 (Output Readout):** We detect the photon at the output using a measurement device. We can choose the basis for this final measurement.

4.3 Scenario 1: Measuring Output in Computational Basis $\{|0\rangle, |1\rangle\}$

First, let us try measuring the final output in the standard $\{|0\rangle, |1\rangle\}$ basis.

Case A: Bomb is Dud The bomb does nothing. The state remains $|\psi\rangle = |+\rangle$. When we measure $|+\rangle$ in $\{|0\rangle, |1\rangle\}$, we get:

- Outcome $|0\rangle$ with probability $|\langle 0|+\rangle|^2 = 1/2$.
- Outcome $|1\rangle$ with probability $|\langle 1|+\rangle|^2 = 1/2$.

Case B: Bomb is Live The bomb performs the first measurement on $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

- **Explosion:** With prob $1/2$, the photon is found on path $|1\rangle$. The bomb explodes.
- **Survival:** With prob $1/2$, the photon is found on path $|0\rangle$. The state collapses to $|0\rangle$.

If it survives (state is now $|0\rangle$), we perform the final measurement in $\{|0\rangle, |1\rangle\}$:

- Outcome $|0\rangle$: Probability 1 (since state is $|0\rangle$).
- Outcome $|1\rangle$: Probability 0.

Conclusion for Scenario 1:

- If we see $|1\rangle$, the bomb must be a Dud.
- If we see $|0\rangle$, it could be a Dud (prob 1/2) or a Live bomb that didn't explode (prob 1/2). This gives us no interaction-free detection of a live bomb.

4.4 Scenario 2: Measuring Output in Hadamard Basis $\{|+\rangle, |-\rangle\}$

Now, we change the output detector to measure in the basis $\{|+\rangle, |-\rangle\}$. This is where the quantum advantage appears.

Case A: Bomb is Dud The bomb does nothing. The state remains $|\psi\rangle = |+\rangle$. We measure $|+\rangle$ in the basis $\{|+\rangle, |-\rangle\}$:

- Outcome $|+\rangle$: Probability $|\langle +|+\rangle|^2 = 1$.
- Outcome $|-\rangle$: Probability $|\langle -|+\rangle|^2 = 0$.

Ideally, interference is perfect, and we never see $|-\rangle$.

Case B: Bomb is Live Stage 1 (The Bomb) occurs first:

- **Explosion:** With prob $1/2$. (Path $|1\rangle$ detected).
- **Survival (Collapse):** With prob $1/2$. The photon is forced to path $|0\rangle$. The state collapses to $|0\rangle$.

Now, we perform the Stage 2 measurement on the surviving state $|0\rangle$. Recall that $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$.

- Outcome $|+\rangle$: Probability $|\langle +|0\rangle|^2 = 1/2$.
- Outcome $|-\rangle$: Probability $|\langle -|0\rangle|^2 = 1/2$.

Total Probabilities for Live Bomb: Combining the probability of survival ($1/2$) with the measurement outcomes:

- **Bomb Explodes:** $1/2$.
- **Outcome $|+\rangle$:** $\frac{1}{2} \times \frac{1}{2} = 1/4$.
- **Outcome $|-\rangle$:** $\frac{1}{2} \times \frac{1}{2} = 1/4$.

4.5 Conclusion: Interaction-Free Detection

Comparing the two cases in Scenario 2, we observe a unique signature:

- If the bomb is a Dud, outcome $|-\rangle$ is **impossible** (Prob 0).
- If the bomb is Live, outcome $|-\rangle$ occurs with probability **1/4**.

Thus, if we observe the outcome $|-\rangle$, we know with certainty that **the bomb is live**, even though the bomb did not explode. The photon must have taken path $|0\rangle$ (avoiding the bomb), yet the *potential* for measurement on path $|1\rangle$ destroyed the interference, making the outcome $|-\rangle$ possible. We have successfully detected the bomb with a probability of 0.25 without interacting with it.

References

[NC10] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, Cambridge, UK, 10th anniversary edition edition, 2010.